m-Polar Picture Fuzzy Ideal of a BCK Algebra

Shovan Dogra, Madhumangal Pal*  
Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, India

ABSTRACT

In this paper, the notions of m-polar picture fuzzy subalgebra (PFSA), m-polar picture fuzzy ideal (PFI) and m-polar picture fuzzy implicative ideal (PFII) of BCK algebra are introduced and some related basic results are presented. A relation between m-polar PFI and m-polar PFII is established. It is shown that an m-polar PFII of a BCK algebra is an m-polar PFI. But the converse of the proposition is not necessarily true. Converse is true only in implicative BCK algebra. The concept of m-polar picture fuzzy commutative ideal (PFCI) is also explored here and some related results are investigated.

© 2020 The Authors. Published by Atlantis Press SARL.  
This is an open access article distributed under the CC BY-NC 4.0 license (http://creativecommons.org/licenses/by-nc/4.0/).

1. INTRODUCTION

After the initiation of fuzzy set (FS) by Zadeh [1] in 1965, the notion of intuitionistic fuzzy set (IFS) was propounded by Atanassov [2] in 1986. IFS includes both the degree of membership (DMS) and the degree of non-membership (DNonMS), whereas fuzzy set includes only the DMS. The concept of BCI/BCK algebra was presented by Iseki and co-workers [3–5]. Merging the concepts of FS and BCK algebra, fuzzy BCK algebra was initiated by Xi [6]. In 1993, the idea of FS was connected with BCI algebra by Ahmad [7]. Later on a lot of works on BCK/BCI algebra and ideals in fuzzy set environment were done by several researchers [8–12]. Intuitionistic fuzzy subalgebra and intuitionistic fuzzy ideal (IFI) in BCK algebra were presented by Jun and Kim [13] in 2000 as an extension of FS concept in BCK algebra. As the time goes, BCK/BCI algebra and ideals were studied by Senapati et al. [14,15] in context of intuitionistic in various directions. Bipolar fuzzy set (BFS) [16] is the generalization of FS which involves the degree of positive membership (DPMS) and the degree of negative membership (DNegMS) of an element. Bipolar fuzzy environment can be realized by an example. The serials broadcasted in Television have both good effect and bad effect on young generation. Good effect can be treated as positive effect and bad effect can be treated as negative effect. Extension work on BFS was given by Chen [17] in the form of m-polar FS. In 2013, including the measure of neutral membership and generalizing the notion of IFS, the concept of picture fuzzy set (PFS) was initiated by Cuong [18]. After the initiation of PFS, different types of research works in context of PFS were performed by several researchers [19–21]. In this paper, we introduce the concept of m-polar picture fuzzy subalgebra (PFSA), m-polar picture fuzzy ideal (PFI) and m-polar picture fuzzy implicative ideal (PFII), m-polar picture fuzzy commutative ideal (PFII) of BCK algebra and explore some results related to these. Also, we develop relationships of m-polar PFI with m-polar PFII and m-polar PFCI of BCK algebra.

2. LIST OF ABBREVIATIONS

| FS | Fuzzy set |
| IFS | Intuitionistic fuzzy set |
| BFS | Bipolar fuzzy Set |
| PFS | Picture fuzzy set |
| DMS | Degree of membership |
| DNonMS | Degree of non-membership |
| DPMS | Degree of positive membership |
| DNegMS | Degree of negative membership |
| DNeuMS | Degree of neutral membership |
| FI | Fuzzy ideal |
| IFI | Intuitionistic fuzzy ideal |
| PFSA | Picture fuzzy subalgebra |
| PFI | Picture fuzzy ideal |

*Corresponding author. Email: mmpalvu@gmail.com
3. PRELIMINARIES

Here, we recapitulate some basic concepts of FS, IFS, BCK/BCI algebra, FI, IFI, BFS, m-polar FS and PFS. We define m-polar PFS, some basic operations on m-polar PFSs, \((\Theta, \phi, \psi)\)-cut of m-polar PFS, image and inverse of m-polar PFS.

**Definition 1.** Let \(A\) be the set of universe. Then a FS [1] \(P\) over \(A\) is defined as \(P = \{(a, \mu_P(a)) : a \in A\}\), where \(\mu_P : A \rightarrow [0, 1]\). Here, \(\mu_P(a)\) is DMS of \(a\) in \(P\).

The DNonMS was missing in FS. Including this type of uncertainty, Atanassov defined IFS in 1986.

**Definition 2.** Let \(A\) be the set of universe. An IFS [2] \(P\) over \(A\) is defined as \(P = \{(a, \mu_P(a), \nu_P(a)) : a \in A\}\), where \(\mu_P(a) \in [0, 1]\) is the DMS of \(a\) in \(P\) and \(\nu_P(a) \in [0, 1]\) is the DNonMS of \(a\) in \(P\) with the condition \(0 \leq \mu_P(a) + \nu_P(a) \leq 1\) for all \(a \in A\).

Here, \(s_P(a) = 1 - (\mu_P(a) + \nu_P(a))\) is the measure of suspicion of \(a\) in \(P\), which excludes the DMS and the DNonMS.

Iseki introduced a special type of algebra namely BCI algebra in 1980.

**Definition 3.** An algebra \((A, \Diamond, 0)\) is said to be BCI algebra [4] if for any \(a, b, c \in A\), the below stated conditions are meet.

i. \([(a \odot b) \odot (a \odot c)] \odot (c \odot b) = 0\)

ii. \([a \odot (a \odot b)] \odot b = 0\)

iii. \(a \odot a = 0\)

iv. \(a \odot b = 0 \land b \odot a = 0 \Rightarrow a = b\)

A BCI algebra with the condition \(0 \odot a = 0\) for all \(a \in A\) is called BCK algebra.

A relation \(\leq\) on \(A\) is defined as \(a \leq b\) iff \(a \odot b = 0\).

**Proposition 1.** In a BCK algebra \((A, \Diamond, 0)\) the following holds.

i. \(0 \odot a = 0\)

ii. \(a \odot 0 = a\)

iii. \(a \odot (a \odot b) \leq b\)

iv. \(a \odot b \leq a\)

v. \((a \odot b) \odot c = (a \odot c) \odot b\)

vi. \((a \odot (a \odot (a \odot b))) = a \odot b\) for all \(a, b, c \in A\)

**Definition 4.** Let \((A, \Diamond, 0)\) be a BCK algebra and \(P = (\mu_P, \eta_P, \nu_P)\) be a FS in \(A\). Then \(P\) is said to be IFI [6] of \(A\) if

i. \(\mu_P(0) \geq \mu_P(a)\)

ii. \(\mu_P(a) \geq \mu_P(a \odot b) \land \mu_P(b)\) for all \(a, b \in A\) and for \(l = 1, 2, ..., m\)

**Definition 5.** Let \((A, \Diamond, 0)\) be a BCK algebra and \(P = (\mu_P, \eta_P, \nu_P)\) be an IFS in \(A\). Then \(P\) is said to be IFI [13] of \(A\) if

i. \(\mu_P(0) \geq \mu_P(a)\) and \(\nu_P(0) \leq \nu_P(a)\)

ii. \(\mu_P(a) \geq \mu_P(a \odot b) \land \mu_P(b)\) and \(\nu_P(a) \leq \nu_P(a \odot b) \lor \nu_P(b)\) for all \(a, b \in A\)

**Definition 6.** A BFS [16] \(P\) is defined as \(P = (a, \mu_P(a), \nu_P(a)) : a \in A\), where \(\mu_P(a) \in [0, 1]\) measures how much a particular property is satisfied by an element and \(\nu_P(a) \in [-1, 0]\) measures how much its anti property is satisfied by that element. DFS 0 means the element has no relevancy to the property.

**Definition 7.** An m-polar FS [17] \(P\) over the set of universe \(A\) is an object of the form \(P = \{(a, \mu_P(a)) : a \in A\}\), where \(\mu_P : A \rightarrow [0, 1]^m\) (\(m\) is a natural number). Here, \([0, 1]^m\) is the poset with respect to partial order relation \(\leq\) which is defined as: \(a \leq b\) for all \(a \in A\) and \(b \in A\) is called \(l\)-th projection mapping.

Including more possible types of uncertainty, Cuong defined PFS in 2013 generalizing the concepts of FS and IFS.

**Definition 8.** Let \(A\) be the set of universe. Then a PFS [18] \(P\) over the universe \(A\) is defined as \(P = \{(a, \mu_P(a), \eta_P(a), \nu_P(a)) : a \in A\}\), where \(\mu_P(a) \in [0, 1]\) is the DPMS of \(a\) in \(P\), \(\eta_P(a) \in [0, 1]\) is the degree of neutral membership (DNeuMS) of \(a\) in \(P\) and \(\nu_P(a) \in [0, 1]\) is the DNegMS of \(a\) in \(P\) with the condition \(0 \leq \mu_P(a) + \eta_P(a) + \nu_P(a) \leq 1\) for all \(a \in A\). For all \(a \in A\), \(1 - (\mu_P(a) + \eta_P(a) + \nu_P(a))\) is the measure of denial membership in \(P\). Sometimes, \((\mu_P(a), \eta_P(a), \nu_P(a))\) is called picture fuzzy value for \(a \in A\).

Motivated by this definition, below we define m-polar PFS.

**Definition 9.** An m-polar PFS \(P\) over the set of universe \(A\) is an object of the form \(P = \{(a, \mu_P(a), \eta_P(a), \nu_P(a)) : a \in A\}\), where \(\mu_P : A \rightarrow [0, 1]^m\), \(\eta_P : A \rightarrow [0, 1]^m\) and \(\nu_P : A \rightarrow [0, 1]^m\) (\(m\) is a natural number) with the condition \(0 \leq \mu_P(a) + \eta_P(a) + \nu_P(a) \leq 1\) for all \(a \in A\) and for \(l = 1, 2, ..., m\). For all \(a \in A\), each of \(\mu_P(a), \eta_P(a)\) and \(\nu_P(a)\) is \(m\)-tuple fuzzy value. Here, \(p_l \circ \mu_P(a), p_l \circ \eta_P(a)\) and \(p_l \circ \nu_P(a)\) represent \(l\)-th components of \(\mu_P(a), \eta_P(a)\) and \(\nu_P(a)\) respectively for \(l = 1, 2, ..., m\).

The basic operations on m-polar PFSs consisting of equality, union and intersection are defined below.

**Definition 10.** Let \(P = \{(a, \mu_P(a), \eta_P(a), \nu_P(a)) : a \in A\}\) and \(Q = \{(a, \mu_Q(a), \eta_Q(a), \nu_Q(a)) : a \in A\}\) be two m-polar PFSs over the universe \(A\). Then

i. \(P \subseteq Q\) if \(p_l \circ \mu_P(a) \leq p_l \circ \mu_Q(a), p_l \circ \eta_P(a) \leq p_l \circ \eta_Q(a)\) and \(p_l \circ \nu_P(a) \geq p_l \circ \nu_Q(a)\) for all \(a \in A\) and for \(l = 1, 2, ..., m\).

ii. \(P = Q\) if \(p_l \circ \mu_P(a) = p_l \circ \mu_Q(a), p_l \circ \eta_P(a) = p_l \circ \eta_Q(a)\) and \(p_l \circ \nu_P(a) = p_l \circ \nu_Q(a)\) for all \(a \in A\) and for \(l = 1, 2, ..., m\).

iii. \(p_l \circ (P \cup Q) = \{(a, \max(p_l \circ \mu_P(a), p_l \circ \mu_Q(a)), \min(p_l \circ \eta_P(a), p_l \circ \eta_Q(a)), \min(p_l \circ \nu_P(a), p_l \circ \nu_Q(a))) : a \in A\}\) for \(l = 1, 2, ..., m\).

iv. \(p_l \circ (P \cap Q) = \{(a, \min(p_l \circ \mu_P(a), p_l \circ \mu_Q(a)), \max(p_l \circ \eta_P(a), p_l \circ \eta_Q(a)), \max(p_l \circ \nu_P(a), p_l \circ \nu_Q(a))) : a \in A\}\) for \(l = 1, 2, ..., m\).

**Definition 11.** Let \(P = \{(a, \mu_P(a), \eta_P(a), \nu_P(a)) : a \in A\}\) be an m-polar PFS over the universe \(A\). Then \((\Theta, \phi, \psi)\)-cut of \(P\) is the crisp set in
A denoted by $C_{a, b, s, t}$ and is defined as $C_{a, b, s, t}(P) = \{ a \in A : p_1 \circ p_2 \geq p_1 \circ \hat{\phi}, p_1 \circ \hat{\eta}(a) \geq p_1 \circ \hat{\phi}, p_1 \circ \psi(a) \leq p_1 \circ \hat{\psi} \text{ for } l = 1, 2, ..., m \}$, where $p_1 \circ \hat{\phi} \in [0, 1], p_1 \circ \hat{\phi} \in [0, 1], p_1 \circ \hat{\psi} \in [0, 1] \text{ with the condition } 0 \leq p_1 \circ \hat{\phi} + p_1 \circ \hat{\phi} + p_1 \circ \hat{\psi} \leq 1 \text{ for } l = 1, 2, ..., m$.

The mentionable fact is that each of $\hat{\phi}, \hat{\psi} \text{ and } p_1 \circ \psi$ represent $l$-th components of the $m$-polar fuzzy values $\hat{\phi}, \hat{\psi} \text{ and } p_1 \circ \psi$ for $l = 1, 2, ..., m$.

**Definition 12.** Let $A_1$ and $A_2$ be two sets of universe. Let $h : A_1 \rightarrow A_2$ be a surjective mapping and $P = \{(a_1, \mu(a_1), \eta(a_1), v(a_1)) : a_1 \in A_1\}$ be an $m$-polar PFS in $A_1$. Then the image of $P$ under the map $h$ is the $m$-polar PFS $h(P) = \{(a_2, \hat{\mu}(a_2), \hat{\eta}(a_2), \hat{v}(a_2)) : a_2 \in A_2\}$, where $p_1 \circ \hat{\mu}(a_2) = \bigvee_{a_1 \in h^{-1}(a_2)} p_1 \circ \mu(a_1) \text{ and } p_1 \circ \hat{\eta}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} p_1 \circ \eta(a_1)$.

**Definition 13.** Let $A_1$ and $A_2$ be two sets of universe. Let $h : A_1 \rightarrow A_2$ be a mapping and $P = \{(a_2, \mu_2(a_2), \eta_2(a_2), v_2(a_2)) : a_2 \in A_2\}$ be an $m$-polar PFS in $A_2$. Then the inverse image of $P$ under the map $h$ is the $m$-polar PFS $h^{-1}(Q) = \{(a_1, \mu_{h^{-1}(Q)}(a_1), \eta_{h^{-1}(Q)}(a_1), v_{h^{-1}(Q)}(a_1)) : a_1 \in A_1\}$, where $p_1 \circ \mu_{h^{-1}(Q)}(a_1) = p_1 \circ \mu(h(a_1)) \text{ and } p_1 \circ \eta_{h^{-1}(Q)}(a_1) = p_1 \circ \eta(h(a_1))$.

**Definition 14.** Let $P = \{(a_1, \mu_1(a_1), \eta_1(a_1), v_1(a_1)) : a_1 \in A_1\}$ and $Q = \{(a_2, \mu_2(a_2), \eta_2(a_2), v_2(a_2)) : a_2 \in A_2\}$ be two $m$-polar PFSs over the universe of $A_1$ and $A_2$ respectively. Then the Cartesian product of $P$ and $Q$ is the $m$-polar PFS $P \times Q = \{(a, b, \mu(a, b), \eta(a, b), v_{\mu}(a, b), v_{\eta}(a, b)) : (a, b) \in A_1 \times A_2\}$, where $p_1 \circ \mu_{\times}(a, b) = p_1 \circ \mu(a) \land p_1 \circ \mu(b), p_1 \circ \eta_{\times}(a, b) = p_1 \circ \eta(a) \land p_1 \circ \eta(b)$.

**Example 1.** Consider a BCK algebra $(A, \circ, 0)$ defined in the following tabular form:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>p</th>
<th>q</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>p</td>
<td>0</td>
<td>0</td>
<td>p</td>
</tr>
<tr>
<td>q</td>
<td>q</td>
<td>q</td>
<td>0</td>
<td>q</td>
</tr>
<tr>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>0</td>
</tr>
</tbody>
</table>

Now, let us consider a 3-polar PFS $P$ as follows:

$$
\mu_p(a) = \begin{cases} 
0.25, 0.35, 0.4, & \text{if } a = 0 \\
0.15, 0.25, 0.35, & \text{if } a = p \\
0.1, 0.15, 0.25, & \text{if } a = q \\
0.15, 0.25, 0.4, & \text{if } a = r 
\end{cases}
$$

and

$$
\eta_p(a) = \begin{cases} 
(0.2, 0.3, 0.4), & \text{if } a = 0 \\
(0.1, 0.2, 0.3), & \text{if } a = p \\
(0.05, 0.1, 0.2), & \text{if } a = q \\
(0.1, 0.2, 0.35), & \text{if } a = r 
\end{cases}
$$

It is easy to show that $P$ is a 3-polar PFSA of $A$.

**Proposition 2.** Let $P = (\mu_p, \eta_p, v_p)$ be an $m$-polar PFSA of a BCK algebra $A$. Then $p_1 \circ \mu_p(a) \geq p_1 \circ \mu_p(b), p_1 \circ \eta_p(a) \geq p_1 \circ \eta_p(b) \text{ and } p_1 \circ v_p(a) \leq p_1 \circ v_p(b) \text{ for all } a, b \in A$ and for $l = 1, 2, ..., m$.

**Proof.** It is observed that

$$
p_1 \circ \mu_p(0) = p_1 \circ \mu_p(a \circ a) \geq p_1 \circ \mu_p(a) \land p_1 \circ \mu_p(a) \text{ [because } P \text{ is an } m\text{-polar PFSA of } A]\n$$

$$
p_1 \circ \eta_p(0) = p_1 \circ \eta_p(a \circ a) \geq p_1 \circ \eta_p(a) \land p_1 \circ \eta_p(a) \text{ [because } P \text{ is an } m\text{-polar PFSA of } A]\n$$

and

$$
p_1 \circ v_P(0) = p_1 \circ v_P(a \circ a) \leq p_1 \circ v_P(a) \land p_1 \circ v_P(a) \text{ [because } P \text{ is an } m\text{-polar PFSA of } A]\n$$

Thus, $p_1 \circ \mu_p(0) \geq p_1 \circ \mu_p(a), p_1 \circ \eta_p(0) \geq p_1 \circ \eta_p(a) \text{ and } p_1 \circ v_p(0) \leq p_1 \circ v_p(a) \text{ for all } a, b \in A \text{ and for } l = 1, 2, ..., m$.

Now, let us define $m$-polar PFI of a BCK algebra.

**Definition 16.** Let $(A, \circ, 0)$ be a BCK algebra and $P = (\mu_p, \eta_p, v_p)$ be an $m$-polar PFS in $A$. Then $P$ is said to be an $m$-polar PFI if (i) $p_1 \circ \mu_p(0) \geq p_1 \circ \mu_p(a), p_1 \circ \eta_p(0) \geq p_1 \circ \eta_p(a) = p_1 \circ v_p(0) \leq p_1 \circ v_p(a)$ and (ii) $p_1 \circ \mu_p(0) \geq p_1 \circ \mu_p(a \circ b) \land p_1 \circ \mu_p(b), p_1 \circ \eta_p(a \circ b) \geq p_1 \circ \eta_p(a \circ b) \land p_1 \circ v_p(0) \leq p_1 \circ v_p(a \circ b) \land p_1 \circ v_p(b) \text{ for all } a, b \in A \text{ and for } l = 1, 2, ..., m$.

Now, we are going to investigate some important results on $m$-polar PFI of a BCK algebra.

**Proposition 3.** Let $P = (\mu_p, \eta_p, v_p)$ be an $m$-polar PFI of a BCK algebra $(A, \circ, 0)$. Then $p_1 \circ \mu_p(a) \geq p_1 \circ \mu_p(b), p_1 \circ \eta_p(a) \geq p_1 \circ \eta_p(b) \text{ and } p_1 \circ v_p(a) \leq p_1 \circ v_p(b) \text{ for all } a, b \in A$ with $a \leq b$ and for $l = 1, 2, ..., m$. 
Proof. Let \( a, b \in A \) such that \( a \leq b \). Then \( a \triangle b = 0 \).

Now, \( p_1 \circ \mu_p(a) \)
\[ \geq p_1 \circ \mu_p(a \triangle b) \land p_1 \circ \mu_p(b) \quad \text{[as \( P \) is an \( m \)-polar PFI of \( A \)]} \]
\[ = p_1 \circ \mu_p(0) \land p_1 \circ \mu_p(b) \]
\[ = p_1 \circ \mu_p(b) \quad \text{[as \( P \) is an \( m \)-polar PFI of \( A \)]} \]
\[ \geq p_1 \circ \eta_p(a) \land p_1 \circ \eta_p(b) \quad \text{[as \( P \) is an \( m \)-polar PFI of \( A \)]} \]
\[ = p_1 \circ \eta_p(0) \land p_1 \circ \eta_p(b) \]
\[ = p_1 \circ \eta_p(b) \quad \text{[as \( P \) is an \( m \)-polar PFI of \( A \)]} \]
and \( p_1 \circ \nu_p(a) \)
\[ \leq p_1 \circ \nu_p(a \triangle b) \lor p_1 \circ \nu_p(b) \quad \text{[as \( P \) is an \( m \)-polar PFI of \( A \)]} \]
\[ = p_1 \circ \nu_p(0) \lor p_1 \circ \nu_p(b) \]
\[ = p_1 \circ \nu_p(b) \quad \text{[as \( P \) is an \( m \)-polar PFI of \( A \)]} \]
for \( l = 1, 2, \ldots, m \).

Thus, \( p_1 \circ \mu_p(a) \geq p_1 \circ \mu_p(b) \leq p_1 \circ \eta_p(a) \land p_1 \circ \eta_p(b) \) and \( p_1 \circ \nu_p(a) \leq p_1 \circ \nu_p(b) \) for \( a, b \in A \) with \( a \leq b \) and for \( l = 1, 2, \ldots, m \).

Proposition 4. Let \((A, \triangle, 0)\) be a BCK algebra and \( P = (\mu_p, \eta_p, \nu_p) \) be an \( m \)-polar PFI of \( A \). Then \( p_1 \circ \mu_p(a) \geq p_1 \circ \mu_p(b) \land p_1 \circ \nu_p(a) \leq p_1 \circ \nu_p(b) \) for all \( a, b \in A \) with \( a \leq b \) and for \( l = 1, 2, \ldots, m \).

Proposition 5. Every \( m \)-polar PFI of a BCK algebra is an \( m \)-polar PFSA.

Proof. Let \((A, \triangle, 0)\) be a BCK algebra and \( A \) is an \( m \)-PFI of \( A \). Since \( P \) is an \( m \)-polar PFI, therefore, \( p_1 \circ \mu_p(a) \geq p_1 \circ \mu_p(b) \land p_1 \circ \nu_p(a) \leq p_1 \circ \nu_p(b) \) for all \( a, b \in A \) with \( a \leq b \). Hence, \( P \) is an \( m \)-polar PFSA of \( A \).

But, the converse of the above proposition is not true in general which is shown in following example. Proposition 6 states under which condition an \( m \)-polar PFSA is an \( m \)-polar PFI.

Example 2. Let us suppose the BCK algebra given in Example 1 and a 3-polar PFS P as follows:

\[ \mu_p(a) = \begin{cases} 0.2, 0.3, 0.4, & \text{if } a = 0, q \\ 0.1, 0.2, 0.3, & \text{if } a = r, \end{cases} \]

\[ \eta_p(a) = \begin{cases} 0.25, 0.35, 0.45, & \text{if } a = 0, q \\ 0.15, 0.25, 0.3, & \text{if } a = r, \end{cases} \]

and \( \nu_p(a) = \begin{cases} 0.3, 0.2, 0.1, & \text{if } a = 0, q \\ 0.4, 0.3, 0.2, & \text{if } a = r, \end{cases} \)

Here, \((0.1, 0.2, 0.3) = \mu_p(p) \geq \mu_p(p \triangle q) \land \mu_p(q) = (0.2, 0.3, 0.4), (0.15, 0.25, 0.3) = \eta_p(p) \geq \eta_p(p \triangle q) \land \eta_p(q) = (0.25, 0.35, 0.45)\) and \((0.4, 0.3, 0.2) = \nu_p(p) \geq \nu_p(p \triangle q) \land \nu_p(q) = (0.3, 0.2, 0.1)\). So, \( P \) is not a 3-polar PFI of \( A \) although it is a 3-polar PFSA.

Proposition 6. Let \((\mu_p, \eta_p, \nu_p)\) be an \( m \)-polar PFSA of a BCK algebra \((A, \triangle, 0)\). Then \( P \) is an \( m \)-polar PFI of \( A \) if for all \( a, b, c \in A \), \( a \triangle b \leq c \Rightarrow p_1 \circ \mu_p(a) \geq p_1 \circ \mu_p(b) \land p_1 \circ \mu_p(c), p_1 \circ \eta_p(a) \geq p_1 \circ \eta_p(b) \land p_1 \circ \eta_p(c) \) and \( p_1 \circ \nu_p(a) \leq p_1 \circ \nu_p(b) \lor p_1 \circ \nu_p(c) \) for \( l = 1, 2, \ldots, m \).

Proof. By given conditions, for all \( a, b, c \in A \), \( a \triangle b \leq c \Rightarrow p_1 \circ \mu_p(a) \geq p_1 \circ \mu_p(b) \land p_1 \circ \mu_p(c) \), \( p_1 \circ \nu_p(a) \leq p_1 \circ \nu_p(b) \lor p_1 \circ \nu_p(c) \) and \( p_1 \circ \nu_p(a) \leq p_1 \circ \nu_p(b) \lor p_1 \circ \nu_p(c) \) for \( l = 1, 2, \ldots, m \).

Thus, \( P \) is an \( m \)-polar PFI of \( A \).
**Proposition 7.** Let \((A, \odot, 0)\) a BCK algebra and \(P = (\mu_P, \eta_P, \nu_P)\), \(Q = (\mu_Q, \eta_Q, \nu_Q)\) be two \(m\)-polar PFIs of \(A\). Then \(P \cap Q\) is an \(m\)-polar PFI of \(A\).

**Proof.** Let \(P \cap Q = R = (\mu_R, \eta_R, \nu_R)\). Then \(p_l \circ \mu_R(a) = p_l \circ \mu_P(a) \land p_l \circ \mu_Q(a) \land p_l \circ \eta_R(a) = p_l \circ \eta_P(a) \land p_l \circ \eta_Q(a)\) and \(p_l \circ \nu_R(a) = p_l \circ \nu_P(a) \lor p_l \circ \nu_Q(a), \forall a \in A\) and for \(l = 1, 2, ... , m\).

Now, \(p_l \circ \mu_R(a)\)
\[= p_l \circ \mu_P(a) \land p_l \circ \mu_Q(a)\]
\[\geq p_l \circ \eta_P(a) \land p_l \circ \eta_Q(a)\]
[as \(P, Q\) are \(m\)-polar PFIs of \(A\)]
\[= p_l \circ \eta_R(a)\]

and \(p_l \circ \nu_R(a)\)
\[= p_l \circ \nu_P(a) \lor p_l \circ \nu_Q(a)\]
\[\leq p_l \circ \nu_P(a) \lor p_l \circ \nu_Q(a)\]
[as \(P, Q\) are \(m\)-polar PFIs of \(A\)]
\[= p_l \circ \nu_R(a), \forall a \in A \text{ and } l = 1, 2, ... , m.\]

Thus, \(a \odot b, b \in C_{\partial, \phi, \psi}(P)\) is a crisp ideal of \(A\).

**Proposition 10.** Let \((A, \odot, 0)\) a BCK algebra and \(P = (\mu_P, \eta_P, \nu_P)\) be an \(m\)-polar PFI in \(A\). Then \(P\) is an \(m\)-polar PFI of \(A\) if all \((\partial, \phi, \psi)\)-cuts of \(P\) are crisp ideals of \(A\).

**Proof.** Let \(a, b \in A\). Then \(p_l \circ \mu_P(a \odot b) \land p_l \circ \mu_P(b) = p_l \circ \theta, p_l \circ \eta_P(a \odot b) \land p_l \circ \eta_P(b) = p_l \circ \phi\) and \(p_l \circ \nu_P(a \odot b) \lor p_l \circ \nu_P(b) = p_l \circ \psi\) for \(l = 1, 2, ... , m\). Clearly, \(p_l \circ \theta \in [0, 1], p_l \circ \phi \in [0, 1]\) and \(p_l \circ \psi \in [0, 1]\) with \(0 \leq p_l \circ \theta + p_l \circ \phi + p_l \circ \psi \leq 1\) for \(l = 1, 2, ... , m\).

Thus, \(a \odot b, b \in C_{\partial, \phi, \psi}(P)\). Since \(C_{\partial, \phi, \psi}(P)\) is a crisp ideal of \(A\) therefore \(a \odot b \in C_{\partial, \phi, \psi}(P)\) and \(b \in C_{\partial, \phi, \psi}(P)\) \(\Rightarrow a \in C_{\partial, \phi, \psi}(P)\).

Therefore, \(p_l \circ \mu_P(a) \geq p_l \circ \theta\) and \(p_l \circ \mu_P(b) \geq p_l \circ \partial\) and \(p_l \circ \eta_P(a) \geq p_l \circ \phi\) and \(p_l \circ \eta_P(b) \geq p_l \circ \partial\) and \(p_l \circ \nu_P(a) \geq p_l \circ \theta\) and \(p_l \circ \nu_P(b) \geq p_l \circ \phi\) and \(p_l \circ \nu_P(a \odot b) \geq p_l \circ \theta\) and \(p_l \circ \nu_P(b) \geq p_l \circ \phi\) for \(l = 1, 2, ... , m\). Hence, \(P\) is an \(m\)-polar PFI of \(A\).

5. **PRE-IMAGE AND IMAGE PFI UNDER HOMOMORPHISM OF BCK ALGEBRA**

In the current section, we explore some properties of \(m\)-polar PFI of BCK algebra under homomorphism of BCK algebra.

**Definition 17.** Let \((A_1, \odot, 0)\) and \((A_2, \ast, 0)\) be two BCK algebras. Then a mapping \(h : A_1 \to A_2\) is said to be homomorphism if \(h(a \odot b) = h(a) \ast h(b)\) for all \(a, b \in A_1\).
It is observed that $h(a) * h(a) = 0$ i.e. $h(a \triangle a) = 0$ i.e. $h(0) = 0$.

**Proposition 11.** Let $(A_1, \triangle, 0)$ and $(A_2, *, 0)$ be two BCK algebras and $Q = (\mu_Q, \eta_Q, v_Q)$ be an m-polar PFI of $A_2$. Then for a BCK algebra homomorphism $h : A_1 \rightarrow A_2$, $h^{-1}(Q)$ is an m-polar PFI of $A_1$.

**Proof.** Let $h^{-1}(Q) = (\mu_{h^{-1}(Q)}, \eta_{h^{-1}(Q)}, v_{h^{-1}(Q)})$, where $\mu_{h^{-1}(Q)} = \mu_Q(h(a))$, $\eta_{h^{-1}(Q)}(a) = \eta_Q(h(a))$ and $v_{h^{-1}(Q)}(a) = v_Q(h(a))$ for all $a \in A_1$.

Now, $p_l \circ \mu_{h^{-1}(Q)}(0) = p_l \circ \mu_Q(h(0)) = p_l \circ \mu_Q(0)$ [as $h(0) = 0$] $

\geq p_l \circ \eta_Q(h(a))$ [because $Q$ is an m-polar PFI of $A_2$] $= p_l \circ \eta_{h^{-1}(Q)}(a)$,

and $p_l \circ v_{h^{-1}(Q)}(0) = p_l \circ v_Q(h(0)) = p_l \circ v_Q(0)$ [as $h(0) = 0$] $\leq p_l \circ v_Q(h(a))$ [because $Q$ is an m-polar PFI of $A_2$] $= p_l \circ v_{h^{-1}(Q)}(a)$ for all $a \in A_1$ and for $l = 1, 2, ..., m$.

Thus, $p_l \circ \mu_{h^{-1}(Q)}(0) \geq p_l \circ \mu_{h^{-1}(Q)}(a), p_l \circ \eta_{h^{-1}(Q)}(0) \geq p_l \circ \eta_{h^{-1}(Q)}(a)$ and $p_l \circ v_{h^{-1}(Q)}(0) \leq p_l \circ v_{h^{-1}(Q)}(a)$ for all $a \in A_1$ and for $l = 1, 2, ..., m$.

Also, $p_l \circ \mu_{h^{-1}(Q)}(a)$ $= p_l \circ \mu_Q(h(a))$ $\geq p_l \circ \mu_Q(h(a) \triangle h(b)) \wedge p_l \circ \mu_Q(h(b))$ [because $Q$ is an m-polar PFI of $A_2$] $= p_l \circ \mu_Q(h(a) \triangle b) \wedge p_l \circ \mu_Q(h(b))$ [because $h$ is a homomorphism] $= p_l \circ \mu_Q(h(a) \triangle b) \wedge p_l \circ \eta_Q(h(b))$ $= p_l \circ \eta_{h^{-1}(Q)}(a \triangle b) \wedge p_l \circ \eta_{h^{-1}(Q)}(b)$, and $p_l \circ v_{h^{-1}(Q)}(a)$ $= p_l \circ v_Q(h(a))$ $\leq p_l \circ v_Q(h(a) \triangle h(b)) \vee p_l \circ v_Q(h(b))$ [because $Q$ is an m-polar PFI of $A_2$] $= p_l \circ v_Q(h(a) \triangle b) \vee p_l \circ v_Q(h(b))$ [because $h$ is a homomorphism] $= p_l \circ v_{h^{-1}(Q)}(a \triangle b) \vee p_l \circ v_{h^{-1}(Q)}(b)$ for all $a, b \in A_1$ and for $l = 1, 2, ..., m$.

Hence, $h^{-1}(Q)$ is an m-polar PFI of $A_1$.

**Proposition 12.** Let $(A_1, \triangle)$ and $(A_2, *)$ be two BCK algebras and $P = (\mu_P, \eta_P, v_P)$ be an m-polar PFI of $A_1$. Then for a bijective homomorphism $h : A_1 \rightarrow A_2$, $h(P)$ is an m-polar PFI of $A_2$.

**Proof.** Let $h(P) = (\mu_{h(P)}, \eta_{h(P)}, v_{h(P)})$. Now, let $b \in A_2$.

Then $p_l \circ \mu_{h(P)}(b) = \bigvee_{a \in h^{-1}(b)} p_l \circ \mu_P(a)$,

$= p_l \circ \eta_{h(P)}(a)$ for all $l = 1, 2, ..., m$.

Since $h$ is bijective therefore $h^{-1}(b)$ must be a singleton set. So, for $b \in A_2$, there exists an unique $a \in A_1$ such that $a = h^{-1}(b)$ i.e. $h(a) = b$. Thus, in this case, $p_l \circ \mu_{h(P)}(b) = p_l \circ \mu_{h(P)}(h(a)) = p_l \circ \mu_P(a)$, $p_l \circ \eta_{h(P)}(b) = p_l \circ \eta_{h(P)}(h(a)) = p_l \circ \eta_P(a)$ and $p_l \circ v_{h(P)}(b) = p_l \circ v_{h(P)}(h(a)) = p_l \circ v_P(a)$ for all $b \in A_2$.

Thus, $p_l \circ \mu_{h(P)}(0) = p_l \circ \mu_{h(P)}(h(0)) = p_l \circ \mu_P(0)$ $\geq p_l \circ \eta_P(a)$, $p_l \circ \eta_{h(P)}(0) = p_l \circ \eta_{h(P)}(h(0)) = p_l \circ \eta_P(0)$ $\geq p_l \circ \eta_P(a)$, $p_l \circ \eta_{h(P)}(b) = p_l \circ \eta_{h(P)}(h(a)) = p_l \circ \eta_P(a)$ and $p_l \circ v_{h(P)}(b) = p_l \circ v_{h(P)}(h(a)) = p_l \circ v_P(a)$ for all $b \in A_2$ and for $l = 1, 2, ..., m$.

Also, $p_l \circ \mu_{h(P)}(b)$ $= p_l \circ \mu_{h(P)}(h(a))$ [where $b = h(a)$ for unique $a \in A_1$] $= p_l \circ \mu_P(a)$ $\geq p_l \circ \eta_P(a)$, $p_l \circ v_{h(P)}(h(c)) = p_l \circ v_{h(P)}(h(h(a))) = p_l \circ v_{h(P)}(h(c))$, [as $P$ is an m-polar PFI of $A_1$] $= p_l \circ v_{h(P)}(b \triangle h(c)) \wedge p_l \circ v_{h(P)}(c)$ [because $h$ is a homomorphism] $= p_l \circ v_{h(P)}(b \triangle h(c)) \wedge p_l \circ v_{h(P)}(c)$, and $p_l \circ \eta_{h(P)}(b) = p_l \circ \eta_{h(P)}(h(a)) = p_l \circ \eta_P(a)$ $\geq p_l \circ \eta_P(a)$, $p_l \circ v_{h(P)}(b) = p_l \circ v_{h(P)}(h(a)) = p_l \circ v_P(a)$ $\leq p_l \circ v_P(a)$, $p_l \circ v_{h(P)}(h(c)) = p_l \circ v_{h(P)}(h(h(a))) = p_l \circ v_{h(P)}(h(c))$, [as $h$ is a homomorphism] $= p_l \circ v_{h(P)}(b \triangle h(c)) \wedge p_l \circ v_{h(P)}(c)$, and $p_l \circ \eta_{h(P)}(b) = p_l \circ \eta_{h(P)}(h(a)) = p_l \circ \eta_P(a)$ $\geq p_l \circ \eta_P(a)$, $p_l \circ v_{h(P)}(b) = p_l \circ v_{h(P)}(h(a)) = p_l \circ v_P(a)$ $\leq p_l \circ v_P(a)$, $p_l \circ v_{h(P)}(h(c)) = p_l \circ v_{h(P)}(h(h(a))) = p_l \circ v_{h(P)}(h(c))$, [as $h$ is a homomorphism] $= p_l \circ v_{h(P)}(b \triangle h(c)) \wedge p_l \circ v_{h(P)}(c)$, and $p_l \circ \eta_{h(P)}(b) = p_l \circ \eta_{h(P)}(h(a)) = p_l \circ \eta_P(a)$ $\geq p_l \circ \eta_P(a)$, $p_l \circ v_{h(P)}(b) = p_l \circ v_{h(P)}(h(a)) = p_l \circ v_P(a)$ $\leq p_l \circ v_P(a)$, $p_l \circ v_{h(P)}(h(c)) = p_l \circ v_{h(P)}(h(h(a))) = p_l \circ v_{h(P)}(h(c))$, [as $h$ is a homomorphism] $= p_l \circ v_{h(P)}(b \triangle h(c)) \wedge p_l \circ v_{h(P)}(c)$.
Thus, \( p_1 \circ \mu_{\mu_0}(b) = p_1 \circ \mu_{\mu_0}(b \circ h(c)) \leq p_1 \circ \mu_{\mu_0}(c) \) for all \( c \in A_1 \). Since \( h \) is bijective, therefore \( h(A_1) = A_2 \). So, for all \( c \in A_1 \), \( h(c) \) can capture all the elements of \( A_2 \). Letting \( d = h(c) \), it is observed that the inequalities hold for all \( d \in A_2 \). Since \( b \) is arbitrary, we therefore obtain that \( p_1 \circ \mu_{\mu_0}(b) \geq p_1 \circ \mu_{\mu_0}(b \circ d) \wedge p_1 \circ \mu_{\mu_0}(d) \), \( p_1 \circ \eta_{\mu_0}(b) \geq p_1 \circ \eta_{\mu_0}(b \circ d) \wedge p_1 \circ \eta_{\mu_0}(d) \), and \( p_1 \circ v_{\mu_0}(b) \leq p_1 \circ v_{\mu_0}(b \circ d) \wedge p_1 \circ v_{\mu_0}(d) \) for all \( b, d \in A_2 \) and for \( l = 1, 2, \ldots, m \). Hence, \( h(P) \) is an \( m \)-polar PFII of \( A_2 \).

### 6. \( m \)-POLAR PFII

The current section introduces the concept of implicative BCK algebra, \( m \)-polar PFII of a BCK algebra and studies some properties related to these. We also investigate a relationship between \( m \)-polar PFII and \( m \)-polar PFII of a BCK algebra.

**Definition 18.** A BCK algebra \((A, \Diamond, 0)\) is said to be implicative if \( a = (a \circ b) \circ a \) for all \( a, b \in A \).

**Proposition 13.** An \( m \)-polar PFII \( P = (\mu_0, \eta_0, v_0) \) in a BCK algebra \((A, \Diamond, 0)\) is said to be \( m \)-polar PFII of \( A \) if the below stated conditions are met.

i. \( p_1 \circ \mu_0(0) \geq p_1 \circ \mu_0(a) \), \( p_1 \circ \eta_0(0) \geq p_1 \circ \eta_0(a) \) and \( p_1 \circ v_0(0) \leq p_1 \circ v_0(a) \)

ii. \( p_1 \circ \mu_0(a) \geq p_1 \circ \mu_0(a \circ (b \circ a)) \circ c \) \( \wedge p_1 \circ \mu_0(c) \), \( p_1 \circ \eta_0(a) \geq p_1 \circ \eta_0(a \circ (b \circ a)) \circ c \) \( \wedge p_1 \circ \eta_0(c) \) and \( p_1 \circ v_0(a) \leq p_1 \circ v_0(a \circ (b \circ a)) \circ c \) \( \wedge p_1 \circ v_0(c) \) for all \( a, b, c \in A \) and for \( l = 1, 2, \ldots, m \).

**Example 3.** Let us consider the BCK algebra \((A, \Diamond)\) as follows:

<table>
<thead>
<tr>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>p</td>
<td>p</td>
<td>p</td>
<td>p</td>
<td>p</td>
<td>p</td>
<td>p</td>
<td>p</td>
<td>p</td>
</tr>
<tr>
<td>q</td>
<td>q</td>
<td>q</td>
<td>q</td>
<td>q</td>
<td>q</td>
<td>q</td>
<td>q</td>
<td>q</td>
<td>q</td>
</tr>
<tr>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
</tr>
<tr>
<td>s</td>
<td>s</td>
<td>s</td>
<td>s</td>
<td>s</td>
<td>s</td>
<td>s</td>
<td>s</td>
<td>s</td>
<td>s</td>
</tr>
</tbody>
</table>

Let us consider a 3-polar PFS \( P = (\mu_p, \eta_p, v_p) \) as follows:

\[
\begin{align*}
\mu_p(a) &= \begin{cases} 
(0.39, 0.41, 0.42), & \text{if } a = 0, p, q \\
(0.25, 0.27, 0.3), & \text{if } a = r, s 
\end{cases} \\
\eta_p(a) &= \begin{cases} 
(0.37, 0.39, 0.4), & \text{if } a = 0, p, q \\
(0.29, 0.33, 0.35), & \text{if } a = r, s 
\end{cases} \\
v_p(a) &= \begin{cases} 
(0.14, 0.17, 0.18), & \text{if } a = 0, p, q \\
(0.3, 0.32, 0.35), & \text{if } a = r, s 
\end{cases}
\end{align*}
\]

and

\[
\pi(p) = \begin{cases} 
(0.37, 0.39, 0.4), & \text{if } a = 0, p, q \\
(0.29, 0.33, 0.35), & \text{if } a = r, s 
\end{cases}
\]

It can be easily shown that \( P \) is a 3-polar PFII of \( A \).

**Proposition 14.** Every \( m \)-polar PFII of a BCK algebra \((A, \Diamond, 0)\) is an \( m \)-polar PFII of \( A \).

**Proof.** Let \( P = (\mu_0, \eta_0, v_0) \) be an \( m \)-polar PFII of \( A \).

Then \( p_1 \circ \mu_0(a) \geq p_1 \circ \mu_0(a \circ (b \circ a)) \circ c \wedge p_1 \circ \mu_0(c) \),

\( p_1 \circ \eta_0(a) \geq p_1 \circ \eta_0(a \circ (b \circ a)) \circ c \wedge p_1 \circ \eta_0(c) \)

and \( p_1 \circ v_0(a) \leq p_1 \circ v_0(a \circ (b \circ a)) \circ c \wedge p_1 \circ v_0(c) \)

for all \( a, b, c \in A \) and for \( l = 1, 2, \ldots, m \).

Setting \( b = a \), it is obtained that

\[
\begin{align*}
\mu_0(a) &= \mu_0(a \circ (a \circ a)) \circ c \wedge \mu_0(c) \\
\eta_0(a) &= \eta_0(a \circ (a \circ a)) \circ c \wedge \eta_0(c) \\
v_0(a) &= v_0(a \circ (a \circ a)) \circ c \wedge v_0(c)
\end{align*}
\]

Therefore, \( P \) is an \( m \)-polar PFII of \( A \).

The above proposition does not hold in reverse direction i.e. an \( m \)-polar PFII of a BCK algebra is not necessarily \( m \)-polar PFII which is clear from the following example. It is necessary to mention that in an implicative BCK algebra, the converse of the above proposition holds which is shown through Proposition 15.

**Example 4.** Now, let us consider a 3-polar PFS \( P = (\mu_p, \eta_p, v_p) \) in BCK algebra \( A \) given in Example 3 as follows:

\[
\begin{align*}
\mu_p(a) &= \begin{cases} 
(0.42, 0.43, 0.45), & \text{if } a = 0, q \\
(0.25, 0.27, 0.3), & \text{if } a = p, r, s 
\end{cases} \\
\eta_p(a) &= \begin{cases} 
(0.3, 0.33, 0.35), & \text{if } a = 0, q \\
(0.15, 0.18, 0.2), & \text{if } a = p, r, s 
\end{cases} \\
v_p(a) &= \begin{cases} 
(0.14, 0.16, 0.2), & \text{if } a = 0, q \\
(0.45, 0.48, 0.5), & \text{if } a = p, r, s 
\end{cases}
\end{align*}
\]
It is clear that \(0.25, 0.27, 0.3) = \mu_p(p) \geq \mu_{\emptyset}(p) \land \mu_{\emptyset}(q) = (0.42, 0.43, 0.45) \land \mu_{\emptyset}(q) = (0.42, 0.43, 0.45) = (0.42, 0.43, 0.45), (0.15, 0.18, 0.2) = \eta_p(p) \leq \eta_{\emptyset}(p) \land \eta_{\emptyset}(q) = (0.3, 0.33, 0.35) \land (0.3, 0.33, 0.35) \land (0.45, 0.48, 0.5) = \varphi_p(p) \leq \varphi_{\emptyset}(p) \land \varphi_{\emptyset}(q) = (0.14, 0.16, 0.2) \vee (0.14, 0.16, 0.2) = (0.14, 0.16, 0.2).\) Thus, \(P\) is not 3-polar PFII although it is a 3-polar PFII of \(A\).

**Proposition 15.** In an implicatve BCK algeba, every m-polar PFII is m-polar PFII.

**Proof.** Let \((A, \circ, 0)\) be an implicatve BCK algebra. Therefore, \(a = (a \circ b) \circ a\) for all \(a, b \in A\). Let \(P = (\mu_p, \eta_p, \varphi_p)\) be an m-polar PFII of \(A\). Then

\[
P \circ \mu_p(a) \geq P \circ \mu_p(a) \land P \circ \eta_p(c) \circ \mu_p(c), P \circ \eta_p(a) \geq P \circ \eta_p(a) \land P \circ \eta_p(c) \land P \circ \varphi_p(c)\]

and \(P \circ \varphi_p(a) \leq P \circ \varphi_p(a) \circ \varphi_p(c) \lor P \circ \varphi_p(c)\)

for all \(a, b, c \in A\) and for \(l = 1, 2, ..., m\).

Thus, \(P\) is an m-polar PFII of \(A\).

**Proposition 16.** Let \((A, \circ, 0)\) be a BCK algebra and \(P = (\mu_p, \eta_p, \varphi_p)\) be an m-polar PFII of \(A\). Then \(C_{\emptyset, \emptyset, \emptyset}(P)\) is an implicatve ideal of \(A\), provided that \(p \circ \mu_p(a) \geq p \circ \mu_p(a) \circ a \circ c \circ p \circ \eta_p(c) \circ p \circ \varphi_p(c) \leq p \circ \psi\) for \(l = 1, 2, ..., m\).

**Proof.** Clearly, \(C_{\emptyset, \emptyset, \emptyset}(P)\) contains at least one element. Let \(((a \circ b) \circ a) \circ c \in C_{\emptyset, \emptyset, \emptyset}(P)\). Then \(P \circ \mu_p([((a \circ b) \circ a) \circ c] \geq P \circ \mu_p(a) \circ p \circ \eta_p(c) \circ p \circ \varphi_p(c) \leq P \circ \psi\) and \(P \circ \mu_p(c) \geq P \circ \psi\)

Thus, \(((a \circ b) \circ a) \circ c \in C_{\emptyset, \emptyset, \emptyset}(P)\). Since \(C_{\emptyset, \emptyset, \emptyset}(P)\) is an implicatve ideal of \(A\).

**Proposition 18.** Let \(S_1\) and \(S_2\) be two ideals of a BCK algebra \((A, \circ, 0)\) such that \(S_1 \subseteq S_2\). If \(S_1\) is implicit then \(S_2\) also.

**Proposition 19.** Let \(P_1\) and \(P_2\) be two m-polar PFIs of a BCK algebra \((A, \circ, 0)\) with \(P_1 \subseteq P_2\). If \(P_1\) is m-polar PFII of \(A\) then \(P_2\) also.

**Proof.** Let \(a \in C_{\emptyset, \emptyset, \emptyset}(P_1)\). Then \(P_1 \circ \mu_p(a) \geq P_1 \circ \mu_p(a) \circ a \circ c \circ P_1 \circ \eta_p(c) \circ P_1 \circ \varphi_p(c) \leq P_1 \circ \psi\), \(P_1 \circ \mu_p(c) \circ P_1 \circ \eta_p(c) \circ P_1 \circ \varphi_p(c) \leq P_1 \circ \psi\) for all \(a, b, c \in A\) and \(l = 1, 2, ..., m\). Thus, \(C_{\emptyset, \emptyset, \emptyset}(P_1) \subseteq C_{\emptyset, \emptyset, \emptyset}(P_2)\). As a result, \(C_{\emptyset, \emptyset, \emptyset}(P_2)\) is an implicatve ideal of \(A\).

**Proposition 20.** Let \(P = (\mu_p, \eta_p, \varphi_p)\) be an m-polar PFII of a BCK algebra \(A\). Then the below stated statements are equivalent.

i. \(P\) is m-polar PFII.

ii. \(P \circ \mu_p(a) \geq P \circ \mu_p(a) \circ (b \circ a),\)

\(P \circ \eta_p(a) \geq P \circ \eta_p(a) \circ (b \circ a),\)

\(P \circ \varphi_p(a) \leq P \circ \varphi_p(a) \circ (b \circ a)\)

for all \(a, b \in A\) and \(l = 1, 2, ..., m\).

iii. \(P \circ \mu_p(a) \geq P \circ \mu_p(a) \circ (b \circ a),\)

\(P \circ \eta_p(a) \geq P \circ \eta_p(a) \circ (b \circ a),\)

\(P \circ \varphi_p(a) \leq P \circ \varphi_p(a) \circ (b \circ a)\)

for all \(a, b \in A\) and \(l = 1, 2, ..., m\).
Proof. (i) ⇒ (ii): Since $P$ is an $m$-polar PFII of $A$, therefore,

$$p_i \circ \mu_p(a) \geq p_i \circ \mu_p((a \odot (b \odot a)) \odot 0) \land p_i \circ \mu_p(0)$$

and for $l = 1, 2, ..., m$. As a result,

$$p_i \circ \eta_p(a) \geq p_i \circ \eta_p((a \odot (b \odot a)) \odot 0) \land p_i \circ \eta_p(0)$$

by Proposition 3.

(iii) ⇒ (i): It is known by Proposition 1 that $a \odot (b \odot a) \leq a$. Then by Proposition 3, $p_i \circ \mu_p(a) \leq p_i \circ \mu_p((a \odot (b \odot a)) \odot 0) \land p_i \circ \eta_p(0)$.

Let us consider the BCK algebra $(A, \odot, 0)$. Then $P$ is said to be an $m$-polar picture fuzzy positive implicational ideal (PFPII) if the below stated conditions are met.

Definition 20. Let $(A, \odot, 0)$ be a BCK algebra and $P = (\mu_p, \eta_p, \nu_p)$ be an $m$-polar FFS in $A$. Then $P$ is said to be an $m$-polar PFPII of $A$ if the following conditions are met:

i. $p_i \circ \mu_p(0) \geq p_i \circ \mu_p(a), p_i \circ \eta_p(0) \geq p_i \circ \eta_p(a)$ and $p_i \circ \nu_p(0) \leq p_i \circ \nu_p(a)$

ii. $p_i \circ \mu_p((a \odot (b \odot a)) \odot 0) \geq p_i \circ \mu_p((a \odot (b \odot a)) \odot 0) \land p_i \circ \mu_p(0)$ for all $a, b \in A$ and for $l = 1, 2, ..., m$.

Example 5. Let us consider the BCK algebra $(A, \odot, 0)$ as follows:

<table>
<thead>
<tr>
<th>$\odot$</th>
<th>0</th>
<th>p</th>
<th>q</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>p</td>
<td>0</td>
<td>0</td>
<td>p</td>
</tr>
<tr>
<td>q</td>
<td>q</td>
<td>p</td>
<td>0</td>
<td>q</td>
</tr>
<tr>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
<td>r</td>
</tr>
</tbody>
</table>

Now, let us suppose a 3-polar FFS $P = (\mu_p, \eta_p, \nu_p)$ defined by

$$\mu_p(a) = \begin{cases} 
0.34, 0.36, 0.37, & \text{if } a = 0 \\
0.28, 0.3, 0.32, & \text{if } a = p \\
0.17, 0.18, 0.18, & \text{if } a = q, r 
\end{cases}$$

$$\eta_p(a) = \begin{cases} 
0.35, 0.36, 0.39, & \text{if } a = 0 \\
0.25, 0.27, 0.3, & \text{if } a = p \\
0.2, 0.23, 0.27, & \text{if } a = q, r 
\end{cases}$$

Clearly, $P$ is a 3-polar PFPII of $A$.

Definition 21. A BCK algebra $(A, \odot, 0)$ is said to be commutative if $b \odot (b \odot a) = a \odot (a \odot b)$ for all $a, b \in A$.

Proposition 23. Every $m$-polar PFPI of a BCK algebra is an $m$-polar FFS.

Proof. Let $P = (\mu_p, \eta_p, \nu_p)$ be an $m$-polar FFS of a BCK algebra $(A, \odot, 0)$.

Now, $(a \odot (0 \odot (0 \odot a))) = (a \odot 0) \odot 0 = a$ [by Proposition 1]
The above proposition is not true in reverse direction which is clear from following example. But the converse of the above proposition holds in commutative BCK algebra which is highlighted through Proposition 24.

**Example 6.** Let us consider a BCK algebra \((A, \odot)\) as follows:

<table>
<thead>
<tr>
<th>(a)</th>
<th>0</th>
<th>(p)</th>
<th>(q)</th>
<th>(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(p)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(q)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(r)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(s)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Now, let us suppose a 3-polar PFS \(P = (\mu_p, \eta_p, \nu_p)\) defined by

\[
\mu_p(a) = \begin{cases} 
(0.4, 0.41, 0.43), & \text{if } a = 0 \\
(0.3, 0.32, 0.33), & \text{if } a = p \\
(0.2, 0.24, 0.27), & \text{if } a = q, r, s
\end{cases}
\]

\[
\eta_p(a) = \begin{cases} 
(0.43, 0.45, 0.47), & \text{if } a = 0 \\
(0.35, 0.36, 0.37), & \text{if } a = p \\
(0.21, 0.22, 0.23), & \text{if } a = q, r, s
\end{cases}
\]

\[
\nu_p(a) = \begin{cases} 
(0.08, 0.09, 0.1), & \text{if } a = 0 \\
(0.27, 0.28, 0.3), & \text{if } a = p \\
(0.45, 0.47, 0.5), & \text{if } a = q, r, s
\end{cases}
\]

Clearly, \(P\) is a 3-polar PFI of \(A\).

It is observed that

\[
\mu_p((q \odot (r \odot (r \odot q)))) = \mu_p(q) = (0.2, 0.24, 0.27), \mu_p(q \odot r) \circ 0 \land \mu_p(0) = (0.4, 0.41, 0.43)
\]

\[
\eta_p((q \odot (r \odot (r \odot q)))) = \eta_p(q) = (0.21, 0.22, 0.23), \eta_p((q \odot r) \circ 0) \land \eta_p(0) = (0.36, 0.45, 0.47)
\]

\[
\nu_p((q \odot (r \odot (r \odot q)))) = \nu_p(q) = (0.45, 0.47, 0.5), \nu_p((q \odot r) \circ 0) \lor \nu_p(0) = (0.08, 0.09, 0.1).
\]

Here, \(\mu_p((q \odot (r \odot (r \odot q)))) \not\leq \mu_p((q \odot r) \circ 0) \land \mu_p(0), \eta_p((q \odot (r \odot (r \odot q)))) \not\leq \eta_p((q \odot r) \circ 0) \land \eta_p(0), \nu_p((q \odot (r \odot (r \odot q)))) \not\leq \nu_p((q \odot r) \circ 0) \lor \nu_p(0)\). Clearly, \(P\) is not a 3-polar PFCI of \(A\).

**Proposition 24.** In a commutative BCK algebra, every m-polar PFI is an m-polar PFCI.

**Proof.** Let \(P = (\mu_p, \eta_p, \nu_p)\) be an m-polar PFI of a commutative BCK algebra \((A, \odot, 0)\). We have, \(\{(a \odot (b \odot (c \odot (b \odot a)))) \odot (a \odot (b \odot a)) \odot c\} \odot c = ((a \odot (b \odot (b \odot a))) \odot (a \odot b)) \odot c)\) by Proposition 1

\[
\leq (a \odot (b \odot (c \odot (b \odot a)))) \odot (a \odot (b \odot a)) \odot c)\) by Proposition 1
\]

\[
= (a \odot (a \odot (b \odot a))) \odot (a \odot b)\) as \(A\) is commutative therefore
\(a \odot (a \odot b)) = (b \odot (b \odot a))\) for all \(a, b \in A\)

\[
= (a \odot b) \odot (a \odot b)\) by Proposition 1
\]

\[
i.e. (a \odot (b \odot (c \odot (b \odot a)))) \odot ((a \odot b) \odot c) \leq c.
\]

Thus, by Proposition 4, it is obtained that \(p_l \odot \mu_p((a \odot (b \odot (c \odot (b \odot a)))) \odot (a \odot b) \odot c) \odot c)\) by Proposition 25 (iii)

\[
= p_l \odot \eta_p((a \odot (b \odot (c \odot (b \odot a)))) \odot (a \odot b) \odot c) \odot c)\) by Proposition 20 (iii),
\]

\[
\leq p_l \odot \nu_p((a \odot (b \odot (c \odot (b \odot a)))) \odot (a \odot b) \odot c) \odot c)\) by Proposition 20 (iii)
\]

Therefore, \(P\) is an m-polar PFFII.
By Proposition 25 (iii) and Proposition 3 we get,

\[ p_l \circ \mu_P(a \circ (b \circ a)) \leq p_l \circ \mu_P((b \circ (b \circ a)) \circ (b \circ (b \circ (b \circ a)))) \]
\[ = p_l \circ \mu_P((a \circ (b \circ (b \circ a))))) [\text{by Proposition 20 (iii)}], \]
\[ \leq p_l \circ \eta_P(a \circ (b \circ (b \circ a))) \]
\[ = p_l \circ \nu_P((b \circ (b \circ a)) \circ (b \circ (b \circ (b \circ a)))) [\text{by Proposition 20 (iii)}] \]
\[ = p_l \circ \nu_P(a \circ (b \circ (b \circ a))) [\text{by Proposition 20 (iii)}]. \]

Therefore, \( P \) is an \( m \)-polar PFNI of \( A \).

Conversely, let \( P \) be both \( m \)-polar PFNI and \( m \)-polar PFPI of \( A \).

Since \((b \circ (b \circ a)) \circ (b \circ a) \leq a \circ (b \circ a)\), by Proposition 3,

\[ p_l \circ \mu_P((b \circ (b \circ a)) \circ (b \circ a)) = p_l \circ \mu_P((b \circ (b \circ a))), \]
\[ p_l \circ \eta_P((b \circ (b \circ a)) \circ (b \circ a)) = p_l \circ \eta_P((b \circ (b \circ a))) \]
\[ \text{and } p_l \circ \nu_P((b \circ (b \circ a)) \circ (b \circ a)) = p_l \circ \nu_P((b \circ (b \circ a))) \]

therefore it is obtained that

\[ p_l \circ \mu_P((a \circ (b \circ a)) \leq p_l \circ \mu_P((b \circ (b \circ a))), \]
\[ p_l \circ \eta_P((a \circ (b \circ a)) \leq p_l \circ \eta_P((b \circ (b \circ a))) \]
\[ \text{and } p_l \circ \nu_P((a \circ (b \circ a)) \geq p_l \circ \nu_P((b \circ (b \circ a))) \]

Also, \( a \circ b \leq a \circ (b \circ a) \). Therefore, by Proposition 3,

\[ p_l \circ \mu_P((a \circ (b \circ a)) \leq p_l \circ \mu_P(a \circ (b \circ a)), \]
\[ p_l \circ \eta_P((a \circ (b \circ a)) \leq p_l \circ \eta_P(a \circ (b \circ a)) \]
\[ \text{and } p_l \circ \nu_P((a \circ (b \circ a)) \geq p_l \circ \nu_P(a \circ (b \circ a)) \]

Since \( P \) is an \( m \)-polar PFNI therefore by Proposition 27,

\[ p_l \circ \mu_P((a \circ b) = p_l \circ \mu_P((a \circ (b \circ a))), \]
\[ p_l \circ \eta_P((a \circ b) = p_l \circ \eta_P(a \circ (b \circ a))) \]
\[ \text{and } p_l \circ \nu_P((a \circ b) = p_l \circ \nu_P(a \circ (b \circ a))) \]

Hence it is obtained that

\[ p_l \circ \mu_P((a \circ (b \circ a)) \leq p_l \circ \mu_P((b \circ (b \circ a))), \]
\[ p_l \circ \eta_P((a \circ (b \circ a)) \leq p_l \circ \eta_P((b \circ (b \circ a))) \]
\[ \text{and } p_l \circ \nu_P((a \circ (b \circ a)) \geq p_l \circ \nu_P((b \circ (b \circ a))) \]

Combining (1) and (4), (2) and (5), (3) and (6) it is obtained that

\[ p_l \circ \mu_P((a \circ (b \circ a)) \leq p_l \circ \mu_P((b \circ (b \circ a))) \]
\[ \leq p_l \circ \eta_P((a \circ (b \circ a)) \leq p_l \circ \eta_P((b \circ (b \circ a))) \]
\[ \text{and } p_l \circ \nu_P((a \circ (b \circ a)) \geq p_l \circ \nu_P((b \circ (b \circ a))) \]

So, by Proposition 20 (ii), \( P \) is an \( m \)-polar PFII of \( A \).

8. CONCLUSION

In this paper, we have initiated the notion of \( m \)-polar PFNI and \( m \)-polar PFII of BCK algebra. We have studied some basic results related to them. We have established a relationship between \( m \)-polar PFNI and \( m \)-polar PFII of a BCK algebra. We have also investigated a relationship between \( m \)-polar PFNI and \( m \)-polar PFII. We have studied some properties of \( m \)-polar PFNI under homomorphism of BCK algebra. It is our hope that our works will help the researchers to study some other types of algebraic structures in context of \( m \)-polar PFNI.

CONFLICT OF INTEREST

Authors declare that they have no conflict of interest.

AUTHORS’ CONTRIBUTIONS

S. Dogra Writing, reviewing and editing. M. Pal reviewing, editing and supervision.

Funding Statement

There is no funding source for this work.

ACKNOWLEDGMENTS

Authors are thankful to the reviewers for their valuable suggestions towards the improvement of the paper.

REFERENCES


